



第三十一屆亞太數學 (APMO) 競賽試題

比賽時間: 2019 年 3 月 12 日

比賽地點: 中央大學、清華大學南大校區、高雄師範大學

注意事項: 本試題的內容僅可由 APMO 官方網站發布。在官方正式發布前, 請勿洩露給任何人, 尤其是經由網路傳遞。

時間限制: 計四小時 (9:30 am ~ 1:30 pm)

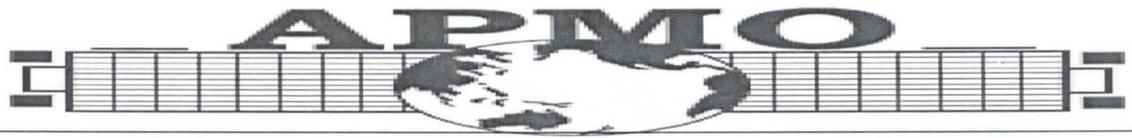
除作圖外, 答案限用黑色或藍色筆書寫。答案不得以修正液 (帶) 修正。

計算紙必須連同試卷交回。不得使用計算器。

本試卷共五題, 每題滿分七分

問題一、 令 \mathbb{Z}^+ 為所有正整數所成的集合。試求所有函數 $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, 使得對於所有正整數 a 和 b , $a^2 + f(a)f(b)$ 都能被 $f(a) + b$ 整除。

Problem 1. Let \mathbb{Z}^+ be the set of positive integers. Determine all functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a and b .



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問題二、給定正整數 m 。無窮數列 $\{a_n\}_{n \geq 1}$ 的定義如下：

a_1 是正整數，且對每一個整數 $n \geq 1$ ，我們有

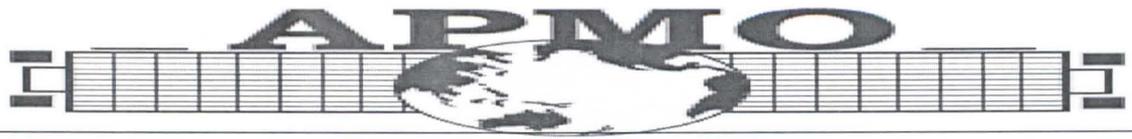
$$a_{n+1} = \begin{cases} a_n^2 + 2^m & \text{若 } a_n < 2^m \\ a_n/2 & \text{若 } a_n \geq 2^m. \end{cases}$$

對每一個 m ，試決定所有可能的 a_1 值，使得數列之每一項都是整數。

Problem 2. Let m be a fixed positive integer. The infinite sequence $\{a_n\}_{n \geq 1}$ is defined in the following way: a_1 is a positive integer, and for every integer $n \geq 1$ we have

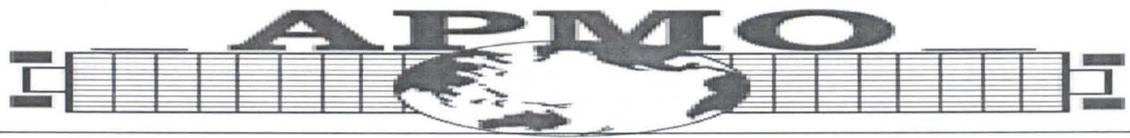
$$a_{n+1} = \begin{cases} a_n^2 + 2^m & \text{if } a_n < 2^m \\ a_n/2 & \text{if } a_n \geq 2^m. \end{cases}$$

For each m , determine all possible values of a_1 such that every term in the sequence is an integer.



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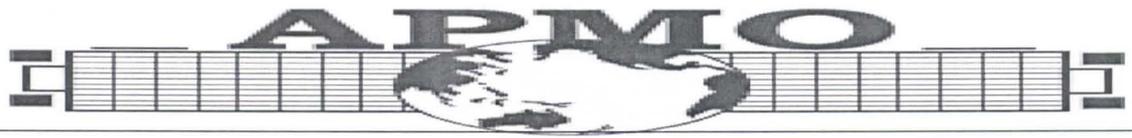


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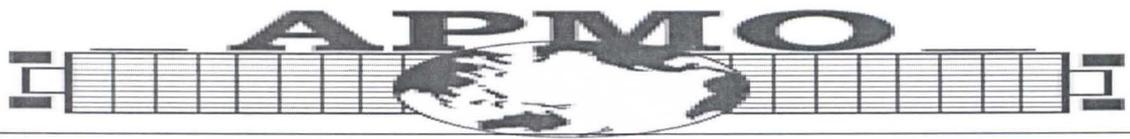


問題三、設三角形 ABC 的任兩邊長均不相等, 且其外接圓為 Γ 。令 BC 的中點為 M 。現選取線段 AM 上的動點 P 。設三角形 BPM 的外接圓與 CPM 的外接圓分別與 Γ 再交於點 D 及 E , 且設直線 DP 與三角形 CPM 的外接圓再交於點 X , 而直線 EP 與三角形 BPM 的外接圓再交於點 Y 。證明存在一個異於點 A 的定點 T , 滿足: 當點 P 變動時, 三角形 AXY 的外接圓必通過點 T 。

Problem 3. Let ABC be a scalene triangle with circumcircle Γ . Let M be the midpoint of BC . A variable point P is selected in the line segment AM . The circumcircles of triangles BPM and CPM intersect Γ again at points D and E , respectively. The lines DP and EP intersect (a second time) the circumcircles to triangles CPM and BPM at X and Y , respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A .



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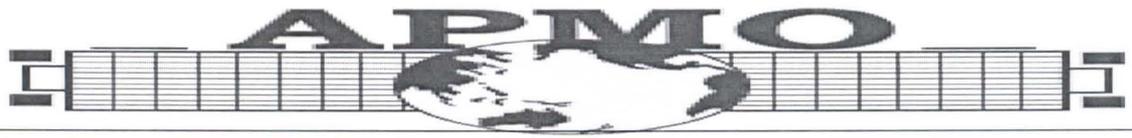


問題四、 考慮一個大小為 2018×2019 的西洋棋棋盤, 其中每一個格子內都有一個整數。如果兩個格子有一條共同邊, 就稱這兩格互為鄰格。在每一回合, 你首先從棋盤上選擇一些格子。接著, 對於被選到的每個格子, 計算該格的所有鄰格的數字平均值。最後, 對於被選到的每個格子, 將其內的數字改為其所有鄰格的數字平均值。試問: 不論一開始棋盤上的數字如何分佈, 是否都可以在有限回合內, 讓所有格子的數字都一樣?

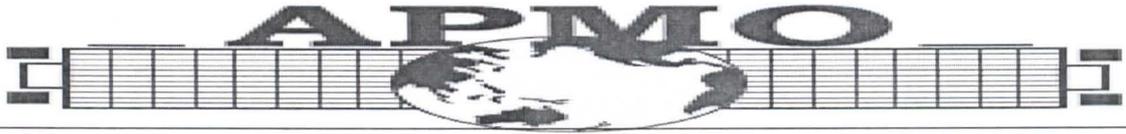
Problem 4. Consider a 2018×2019 board with integers in each unit square. Two unit squares are said to be neighbours if they share a common edge. In each turn, you choose some unit squares. Then for each chosen unit square the average of all its neighbours is calculated. Finally, after these calculations are done, the number in each chosen unit square is replaced by the corresponding average. Is it always possible to make the numbers in all squares become the same after finitely many turns?



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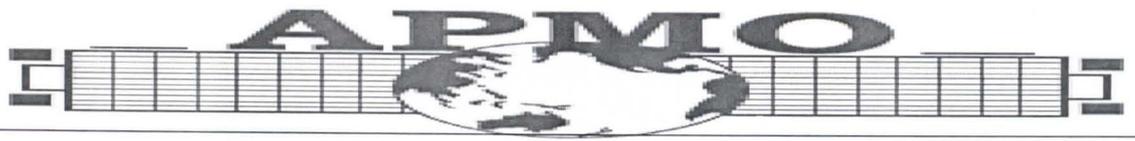
問題五、試決定所有的函數 $f : \mathbb{R} \rightarrow \mathbb{R}$, 使得對所有實數 x 與 y , 均滿足

$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy).$$

Problem 5. Determine all the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy).$$

for all real number x and y .



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