2022 年亞太數學奧林匹亞競賽 初選考試（一）試題

考試時間：2021 年 11 月 13 日上午 10:00 ～ 12:00

說明：本試題共兩頁，分成兩部分：選填題與非選擇題。

作答方式：

- 選填題用 2B 鉛筆在「答案卡」上作答；更正時，應以橡皮擦擦拭，切勿使用修正液（帶）。
- 非選擇題用藍、黑色原子筆在「答案卷」上作答；更正時，可以使用修正液（帶）。
- 未依規定畫記答案卡，致機器掃描無法辨識答案，或未使用藍、黑色原子筆書寫答案卷，致評閱人員無法辨認答案者，其後果由考生自行承擔。
- 不得使用量角器、計算器及其他電子設備。
- 答案卷每人一張，不得要求增補。

第一部分：選填題

說明：本部分共有五題，每題 7 分，答錯不倒扣，未完全答對不給分。

答案卡填答注意事項：答案的數字位數少於填答空格數時，請適當地在前面填入 0。

1. 買一張大樂透時，你必須從 $\{1,2,\cdots,49\}$ 中選出六個相異正整數填在彩券上。每期開獎時，會從 $\{1,2,\cdots,49\}$ 抽出六個數字當獎號，再抽另一個數字當特別號。如果一張彩券上恰有四個數字是獎號，且再有一個數字是特別號，則該張彩券可以得到肆獎。在此規則下，一共有 ①②③ 種數字組合可以得到肆獎。

2. 費波納契數列 F_n 的定義是

   - $F_0 = 0$;
   - $F_1 = 1$;
   - $F_n = F_{n-1} + F_{n-2}$ 對 $n \geq 2$ 時均成立。

   如果 F_n 除以 12 的餘數為 7，則我們稱 n 為費波納契之馬（因十二生肖中的第七位是馬）。

   在小於 1000 的正整數中，共有 ④⑤⑥ 匹費波納契之馬。

3. 考慮像下圖一樣的數值三角形，在兩條邊上依序填入 $0,1,2,\ldots$，而三角形內部的數字則跟巴斯卡三角形一樣，是上列相鄰的兩數字的和。

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 1 & & 1 & & & & & \\
 & & 2 & & 2 & & 2 & & & & \\
 & 3 & & 4 & & 4 & & 3 & & & \\
4 & & 7 & & 8 & & 7 & & 4 & & \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}$$

所以第一列所有數字的總和是 0, 第二列所有數字的總和是 2, 第三列所有數字的總和是 6。而第 2022 列的總和除以 1000 的餘數是 ⑦⑧⑨.

4. 設某個凸四邊形有外接圓，且其四邊長依序分別為 $1,2,3,4$。則此四邊形的外接圓半徑為 $\dfrac{\sqrt{⑩⑪⑫⑬}}{⑭⑮}$。(化為最簡根式)

5. 某國有 110 座城市，其中任 4 座城市之中都至少有 1 座城市，會與這 4 座城市中的其他 3 座城市有鐵路相通。則該國至少有 ⑯⑰⑱ 座城市，其與該國中的所有其他城市之間有鐵路相通。

第二部分：非選擇題

說明：每題 7 分。答案必須寫在「答案卷」上，並標明題號，同時必須寫出演算過程或理由，否則將予扣分甚至零分。作答使用藍、黑色原子筆書寫，除幾何作圖外不得使用鉛筆。

一、 設 $ABCD$ 為凸四邊形，其中直線 AB 與 CD 不平行，同時令點 P 為兩對角線 AC, BD 的交點。在 PD 線段中取一點 M、在 PC 線段中取一點 N 使得 $PM = PN$。若 $\angle CAB = \angle CDB$，試證：直線 MN 與直線 AB 的夾角，等於直線 MN 與直線 CD 的夾角。

二、 已知非負實數 a,b,c,d 滿足 $a+2b+3c+4d=1$ 且 $(a+b)(c+d)^2=(a+c)(b+d)^2=k$. 則 k 的最大值為何？又該最大值在何時發生？

2022 APMO Taiwan Preliminary Round 1

10:00–12:00, November 13, 2021

General instructions.

- There are 3 pages of problems, consisting of fill-in problems and non-multiple-choice problems.
- Use 2B pencils to answer fill-in problems on the designated card. Use erasers only to make corrections for these, do not use correction tape/fluid.
- Use pens in blue or black ink to answer non-multiple-choice problems on the designated sheet of paper. Correction tape/fluid may be used to make corrections for this part.
- Contestants are held responsible for the consequences from failing to follow the instructions above so that the machine cannot read the designated card, or the answers for non-multiple-choice problems are illegible.
- Protractors, calculators and other electronic devices are prohibited.
- One sheet of paper for the non-multiple-choice problems is given to each contestants. No more supply is offered.

Part 1. Fill-in problems

Instruction. There are FIVE problems in this Part. Each problem is worth 7 points. There is no penalty for wrong answers. No marks will be awarded for answers that are not completely correct.

If the number of digits for the answers is less than the number of designated spaces, fill in a proper number of 0's at the beginning of your answer.

1. In the game of Major Lottery, six distinct numbers from the integers 1, 2, …, 49 should be marked on each ticket. The lottery authority will also draw six numbers as Winning Numbers from the integers 1 through 49, and then draw a different number as the Major Number from the same range. A ticket wins the Fourth Prize if there are exactly 4 Winning Numbers and the Major Number on that ticket. Under these rules, there are $\underline{①②③}$ different tickets that win the Fourth Prize.

2. The Fibonacci sequence F_n is defined as follows.

   - $F_0 = 0$;
   - $F_1 = 1$;
   - $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geqslant 2$

   An integer n is called the *Horse of Fibonacci* if the remainder of F_n is 7 when divided by 12 (because Horse is the seventh symbol in the Twelve Zodiacs)

   Among the positive integers less than 1000, there are $\underline{\textcircled{4}\textcircled{5}\textcircled{6}}$ Horses of Fibonacci.

3. Consider the numerical triangle shown below: write $0, 1, 2, \ldots$ in order at the two edges; the numbers in the interior of the triangle follows the same rule as in the Pascal triangle, that is, each number is the sum of the two numbers above it.

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & 1 & & 1 & & & & & \\
 & & 2 & & 2 & & 2 & & & & \\
 & 3 & & 4 & & 4 & & 3 & & & \\
4 & & 7 & & 8 & & 7 & & 4 & & \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}$$

   Accordingly, the sum of all numbers in the first row is 0, the sum of those in the second row is 2, while the sum of those in the third row is 6. The sum of all numbers in the 2022^{nd} row has remainder $\underline{\textcircled{7}\textcircled{8}\textcircled{9}}$ when divided by 1000.

4. The sides of a convex cyclic quadrilateral are $1, 2, 3$, and 4 in this order. The circumradius of this quadrilateral is $\dfrac{\sqrt{\textcircled{10}\textcircled{11}\textcircled{12}\textcircled{13}}}{\textcircled{14}\textcircled{15}}$ (in the reduced form).

5. There are 110 cities in the Wonderland. Among any 4 cities, there is at least one city that connects to the other 3 cities by direct railroad. There are at least $\underline{\textcircled{16}\textcircled{17}\textcircled{18}}$ cities, each of which connects to any other city in the Wonderland by direct railroad.

Part 2. Non-multiple-choice problems

Instruction. There are TWO problems in this part. Each problem is worth 7 points. Answers should be written in blue or black ink, except for graphics that can be drawn by pencil. The problem number should be indicated clearly. The intermediate steps and reasons should be clearly stated, or penalty in deduction of points will be incurred.

I. Let $ABCD$ be a convex quadrilateral, in which line AB and CD are not parallel to each other. Let P be the intersection points of the two diagonals AC and BD. Choose points M and N from segment PD, PC respectively so that $PM = PN$. If $\angle CAB = \angle CDB$, prove that line MN intersects with line AB and CD respectively at the same angle.

II. Non-negative real numbers a, b, c, d satisfy $a + 2b + 3c + 4d = 1$ and $(a+b)(c+d)^2 = (a+c)(b+d)^2 = k$. What is the maximum value for k, and when does this maximum occur?

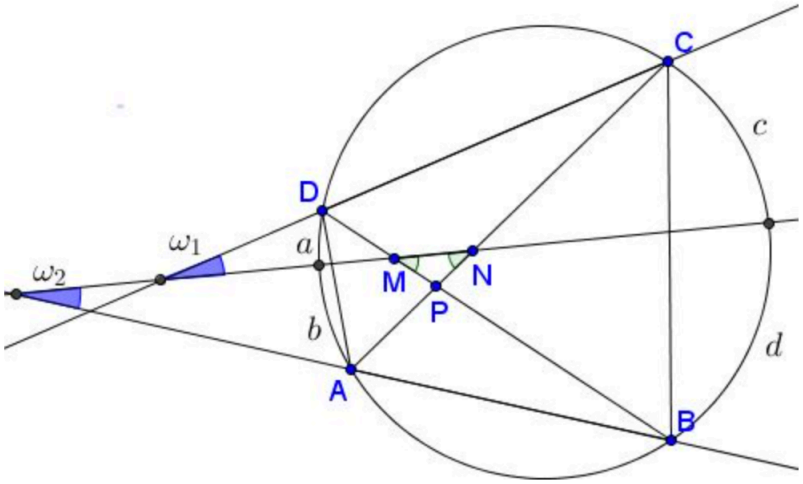
參考答案：

1. 630
2. 042
3. 302
4. $\frac{\sqrt{2310}}{24}$
5. 107

2022 年亞太數學奧林匹亞競賽 初選考試（一）

非選擇題參考解答

一、 **參考解答.** 因為 $\angle CAB = \angle CDB$，所以 $ABCD$ 四點共圓。令 a, b, c, d 分別代表下圖中外接圓的圓弧。



因為 $PM = PN$，所以等腰三角形中 $\angle NMP = \angle PNM$。由圓內兩割線的夾角關係，知

$$\angle NMP = \frac{1}{2}(b+c),$$
$$\angle PNM = \frac{1}{2}(a+d).$$

聯立得 $b+c = a+d$。另一方面，

$$\omega_1 = \frac{1}{2}(c-a),$$
$$\omega_2 = \frac{1}{2}(d-b).$$

故 $\omega_1 = \omega_2$ 等價於 $c-a = d-b$，即 $b+c = a+d$。得證。

二、 **參考解答.** 顯然 k 的最大值必為正數。不失一般性，我們只考慮 $(a+b)(a+c)>0$ 的情況。由 $(a+b)(c+d)^2=(a+c)(b+d)^2=k$ 可整理得到

$$(b-c)\left(b+c+2d-\frac{k}{(a+b)(a+c)}\right)=0,$$

即

$$b=c \quad \text{or} \quad b+c+2d=\frac{k}{(a+b)(a+c)}. \tag{*}$$

(此部分獨立得 1 分。)

Case 1. 若 $b=c$, 我們有 $(a+b)(b+d)^2=k$ 再由算幾不等式可得到

$$1=a+5b+4d=(a+b)+4(b+d)\geq 3(4(a+b)(c+d)^2)^{1/3}=3(4k)^{1/3}.$$

因此，k 的最大值為 $1/108$ ，且等號成立若且唯若 $a+b=2(b+d)=1/3$。在此情況下僅當 $b=c\in[0,1/6]$，$a=1/3-b, d=1/6-b$ 時，k 可達到最大值 $1/108$.

Case 2. 若 $b\neq c$, 沒必要從 (*) 的另一個可能性來處理。我們只需考慮 $b>c, b<c$ 兩種情況即可。同 Case 1 的方法，若 $b>c$, 我們可以得到 $1>(a+b)+4(c+d)\geq 3(4k)^{1/3}$, 證得 $k<1/108$. 同理，當 $b<c$ 時，由 $(a+c)(b+d)^2=k$ 我們可證得 $k<1/108$. 此時 k 不可能達到最大值。